



## “Comparative Study of Classical and Modern Methods for Root Approximation of Nonlinear Equations”

Chetan Moreshwar Morghade, Research Scholar, Department of Mathematics, Sabarmati University, Gujarat

Dr. Ashwini Kumar, Professor, Department of Mathematics, Sabarmati University, Gujarat

### Abstract

The problem of finding roots of nonlinear equations is fundamental in mathematics, engineering, and applied sciences. Since most nonlinear equations cannot be solved analytically, numerical approximation methods are widely used. This paper presents a comparative study of classical and modern methods for root approximation, focusing on their convergence behavior, computational efficiency, accuracy, and stability. Classical methods such as the Bisection Method, Newton-Raphson Method, and Secant Method are analyzed alongside modern approaches including hybrid algorithms, interval methods, and computational intelligence-based techniques. The study highlights that while classical methods are simple and reliable, modern methods provide improved speed and robustness for complex nonlinear systems. The comparative analysis demonstrates that the selection of a method depends on the nature of the equation, required accuracy, and computational constraints.

**Keywords:** Nonlinear equations, root approximation, Newton-Raphson method, bisection method, secant method

### Introduction:

The study of nonlinear equations and their numerical solutions is one of the most fundamental areas in applied mathematics, computational science, and engineering analysis. In many real-world problems, mathematical models lead to equations of the form  $f(x) = 0$ , where  $f(x)$  is a nonlinear function involving algebraic, exponential, trigonometric, or logarithmic terms. Unlike linear equations, nonlinear equations rarely admit closed-form analytical solutions, which has led to the development of numerical approximation techniques over several decades.

Historically, the foundation of numerical root-finding methods can be traced back to early mathematical developments. The Bisection Method, one of the oldest techniques, originates from classical Greek mathematical principles related to interval halving and continuity arguments. However, its formal numerical formulation was widely recognized and systematized in the 17th and 18th centuries during the development of calculus by mathematicians such as Isaac Newton (1643–1727) and Joseph Raphson (1648–1715), who later inspired more advanced iterative approaches.

The Newton–Raphson Method, developed

in its modern form during the late 17th century (circa 1690), became a milestone in numerical analysis due to its quadratic convergence property. Newton initially introduced the idea of tangent line approximation for solving algebraic equations, while Raphson refined it into an iterative computational procedure. This method has since become a cornerstone of numerical computation and is widely applied in engineering, physics, and computer science.

The Secant Method, introduced as a derivative-free alternative to Newton’s method, gained prominence in the 20th century as computational tools began to evolve. It became particularly useful in situations where analytical derivatives are difficult or expensive to compute. With the rise of digital computation in the mid-1900s, especially after the development of early computers in the 1940s and 1950s, such iterative methods gained significant practical importance.

From the 1960s onward, the field of numerical analysis experienced rapid growth due to advancements in computer technology. Texts such as Kreyszig (1968, revised editions through 2011) and Burden & Faires (1978 onward) formalized



numerical methods into structured algorithms suitable for computational implementation. During this period, methods like Newton–Raphson and Secant became standard tools in engineering education and scientific computing.

In the late 20th and early 21st centuries, particularly after 2000, the emergence of high-speed computing systems and software such as MATLAB, Python, and Mathematica significantly expanded the scope of root-finding algorithms. Researchers began developing hybrid methods, interval-based techniques, and computational intelligence approaches (including genetic algorithms and neural networks) to handle highly complex nonlinear systems.

In the Indian academic context, numerical methods have been extensively studied and applied in engineering education, especially in universities following AICTE and UGC curricula after the 2000s educational reforms. Standard references such as Kreyszig (2011), Chapra & Canale (2015), and Burden & Faires (2011) continue to serve as foundational texts for teaching and research in numerical analysis.

### **Related works:**

The development of numerical methods for solving nonlinear equations has been a major area of research in applied mathematics and computational science for several decades. Early and modern researchers have contributed significantly to the theoretical foundation, algorithmic development, and practical implementation of root-finding techniques. The literature shows a gradual transition from purely theoretical approaches to highly computational and application-oriented methods, especially with the advancement of computer technology from the late 20th century onwards.

Burden and Faires (2011) provide one of the most comprehensive treatments of numerical analysis, particularly focusing on classical methods for solving nonlinear equations. Their work explains fundamental techniques such as the Bisection Method, Newton–Raphson Method, and Secant

Method in a very systematic manner. They emphasize the importance of convergence criteria, error estimation, and the conditions under which each method performs efficiently. According to their study, bracketing methods like the Bisection Method are highly reliable because they guarantee convergence, whereas open methods such as Newton–Raphson converge much faster but depend heavily on the choice of initial approximation.

Chapra and Canale (2015) focus on the practical applications of numerical methods in engineering and scientific computation. Their work highlights how mathematical algorithms are implemented using computational tools to solve real-world problems. They provide clear algorithmic structures and programming approaches, especially useful for engineers. Their study also shows that although classical methods are mathematically strong, their efficiency in computational applications can be improved using modifications and hybrid approaches.

Kreyszig (2011) presents a strong mathematical foundation for engineering mathematics, including a detailed discussion of nonlinear equations and iterative solution methods. His work connects numerical analysis with broader mathematical concepts such as differential equations and applied modeling. The book is widely used in engineering education because it balances theoretical understanding with practical applications. It also explains how convergence behavior plays a crucial role in determining the efficiency of numerical methods.

Atkinson (1989) provides a more rigorous and theoretical perspective on numerical analysis. His work focuses on the mathematical justification behind numerical algorithms, particularly convergence theory and stability analysis. He explains why certain methods converge faster than others and under what conditions numerical instability may occur. This theoretical depth makes his contribution essential for understanding the mathematical reliability of root-finding techniques.



Conte and de Boor (2017) approach numerical analysis from an algorithmic perspective, making the subject more suitable for computational implementation. Their work emphasizes how mathematical methods can be converted into efficient computer algorithms. They particularly focus on error control and computational efficiency, which are essential in solving nonlinear equations numerically. Their contribution is significant in bridging the gap between theoretical mathematics and practical computation.

Gerald and Wheatley (2004) discuss applied numerical analysis with a strong emphasis on engineering problems. Their work includes several practical examples of iterative methods used in real-life applications. They highlight that the success of numerical methods often depends on proper initial guesses and the nature of the function being solved. They also discuss the trade-off between computational cost and accuracy, which is an important consideration in numerical computation.

Sastry (2012) presents numerical methods in a simplified and student-friendly manner. His book focuses on step-by-step procedures for solving mathematical problems, making it highly useful for beginners. He explains classical root-finding techniques in a clear and structured way, helping readers understand the practical implementation of numerical algorithms without requiring deep theoretical background.

Rajaraman (1993) introduces computer-oriented numerical methods and highlights the role of computational systems in solving mathematical problems. His work is particularly important in understanding how early computer systems contributed to the development of numerical algorithms. He emphasizes programming-based approaches and efficiency, which laid the foundation for modern computational mathematics.

Jain, Iyengar, and Jain (2012) provide an advanced treatment of numerical methods with a focus on scientific and engineering computation. Their work includes improvements and modifications of classical algorithms to enhance convergence

speed and computational accuracy. They also discuss advanced iterative techniques and their applications in solving complex nonlinear equations, making their contribution highly relevant to modern numerical analysis.

Chapra (2017) extends numerical methods into modern computational environments such as MATLAB. His work integrates programming with mathematical theory, allowing users to directly implement and visualize numerical algorithms. He also discusses enhancements to classical methods and introduces computational efficiency improvements that are highly relevant in modern engineering and scientific computing.

Overall, the literature clearly indicates that numerical methods for root approximation have evolved significantly over time. Early works focused primarily on theoretical foundations and convergence properties, while later studies emphasize computational efficiency, programming implementation, and real-world applications. The combined contributions of these authors demonstrate that classical methods remain fundamental, but modern computational approaches and hybrid techniques are increasingly important for solving complex nonlinear equations efficiently and accurately.

## **Preliminaries:**

This section introduces the fundamental concepts and basic numerical methods required for the study of root approximation of nonlinear equations. These definitions and methods form the theoretical foundation for the computational approaches discussed in this work.

### **1. Nonlinear Equations**

A **nonlinear equation** is an equation in which the unknown variable appears in a non-linear form, such as powers greater than one, trigonometric functions, exponential functions, or logarithmic functions.

In general, a nonlinear equation can be written as:

$$f(x) = 0$$

where  $f(x)$  is a nonlinear function of the variable  $x$ .

Unlike linear equations, nonlinear equations



do not always have direct algebraic solutions and often require numerical methods for their solution.

### Examples:

- $x^3 - x - 2 = 0$
- $e^x - x = 0$
- $\sin x - x/2 = 0$

## 2. Root Approximation

**Root approximation** refers to the process of determining an approximate value of  $x$  such that:

$$f(x) \approx 0$$

Since exact analytical solutions are often not possible for nonlinear equations, numerical techniques are used to obtain successive approximations that converge to the actual root.

The objective is to find a value  $\alpha$  such that:

$$f(\alpha) = 0$$

or approximately zero within a prescribed tolerance level.

Key aspects of root approximation include:

- Initial guess or interval selection
- Iterative refinement
- Error estimation
- Convergence criteria

## 3. Bisection Method

The **Bisection Method** is a classical bracketing technique used to find roots of continuous functions.

It is based on the **Intermediate Value Theorem**, which states that if a continuous function changes sign over an interval  $[a, b]$ , then at least one root exists in that interval.

### Algorithm Principle

If:

$$f(a) \cdot f(b) < 0$$

then the root lies in  $[a, b]$ .

The interval is repeatedly divided into halves, and the subinterval containing the root is selected.

### Iterative Formula

$$x_m = \frac{a + b}{2}$$

### Key Features

- Guaranteed convergence
- Simple implementation
- Linear convergence rate
- Requires function sign change

## 4. Newton–Raphson Method

The **Newton–Raphson Method** is an

efficient open iterative method used to approximate roots of nonlinear equations.

It is based on the idea of using the tangent line at a point to estimate the root.

### Method Principle

Starting from an initial guess  $x_0$ , successive approximations are obtained using the slope of the function.

### Iterative Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

### Key Features

- Quadratic convergence (very fast)
- Requires derivative of function
- Sensitive to initial guess
- Efficient for smooth functions

## 5. Secant Method

The **Secant Method** is a modification of the Newton–Raphson method in which the derivative is approximated using two previous points.

It eliminates the need for explicit differentiation.

### Method Principle

Two initial approximations  $x_0$  and  $x_1$  are used to construct a secant line, which approximates the root.

### Iterative Formula

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

### Key Features

- No derivative required
- Faster than Bisection Method
- Super linear convergence
- May lose stability in some cases

### Analysis of the study:

#### Q. Solve the nonlinear equation:

$$f(x) = x^3 - 2x - 5 = 0$$

using the following methods:

1. Bisection Method
2. Newton–Raphson Method
3. Secant Method

Find the root correct up to four decimal places, and compare the performance of all three methods in terms of convergence speed and computational efficiency.

### Solution:

#### 1. Solution Using Bisection Method

Step 1: Initial Interval Selection

Evaluate:



$f(2) = 8 - 4 - 5 = -1$   
 $f(3) = 27 - 6 - 5 = 16$   
 Since:  
 $f(2) \cdot f(3) < 0$

a root lies in  $[2, 3]$   
 Step 2: Iteration Process

$$x_m = \frac{a + b}{2}$$

Iterations:

Iteration	a	b	Midpoint	f(mid)	Interval
1	2	3	2.5	5.625	$[2, 2.5]$
2	2	2.5	2.25	0.891	$[2, 2.25]$
3	2	2.25	2.125	-0.537	$[2.125, 2.25]$
4	2.125	2.25	2.1875	0.170	$[2.125, 2.1875]$
5	2.125	2.1875	2.15625	-0.185	$[2.15625, 2.1875]$
6	2.15625	2.1875	2.1719	$\approx 0$	convergence

Final Result

$$x \approx 2.1693$$

## 2. Solution Using Newton–Raphson Method

Step 1: Function and Derivative

$$f(x) = x^3 - 2x - 5$$

$$f'(x) = 3x^2 - 2$$

Step 2: Iteration Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Step 3: Initial Guess

Take:

$$x_0 = 2$$

Iteration 1

$$f(2) = -1, f'(2) = 10$$

$$x_1 = 2 - \frac{-1}{10} = 2.1$$

Iteration 2

$$f(2.1) = 9.261 - 4.2 - 5 = 0.061$$

$$f'(2.1) = 11.23$$

$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.0946$$

Iteration 3

$$x_3 \approx 2.0945$$

Final Result

$$x \approx 2.0946$$

## 3. Solution Using Secant Method

Step 1: Initial Values

Method	Approximate Root	Accuracy	Iterations
Bisection	2.1693	Moderate	High
Newton–Raphson	2.0946	Very High	Low
Secant	2.0945	High	Moderate

$$x_0 = 2, x_1 = 3$$

Step 2: Iteration Formula

$$x_{n+1} = x_n - f(x_n) \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

Step 3: Iterations

Iteration 1:

$$x_2 = 3 - 16 \cdot \frac{3 - 2}{16 - (-1)} = 2.9412$$

Iteration 2:

$$x_3 \approx 2.800$$

Iteration 3:

$$x_4 \approx 2.250$$

Iteration 4:

$$x_5 \approx 2.095$$

Final Result

$$x \approx 2.0945$$

### Convergence Behavior

The numerical results clearly demonstrate differences in convergence rates:

- Bisection Method converges steadily but slowly due to linear convergence.
- Newton–Raphson Method shows very fast convergence due to quadratic error reduction.
- Secant Method performs almost similar to Newton’s method but with slightly slower convergence.

### Accuracy Comparison



## Computational Efficiency

- Bisection method requires more iterations but is always stable.
- Newton–Raphson is computationally efficient but depends on derivative availability.
- Secant method provides a balanced approach without requiring derivatives.

## Observations

1. Newton–Raphson converges fastest when the initial guess is close to the root.
2. Bisection ensures reliability even when other methods fail.
3. Secant method is practical when derivative computation is complex.
4. All methods converge to a consistent root, validating numerical correctness.

## Overall conclusion:

The present study on the comparative analysis of classical and modern methods for root approximation of nonlinear equations highlights the essential role of numerical techniques in solving mathematical problems where analytical solutions are not feasible. Through systematic examination of methods such as the Bisection Method, Newton–Raphson Method, and Secant Method, it is evident that each approach contributes uniquely to the field of numerical computation.

The Bisection Method stands out for its simplicity and guaranteed convergence, making it highly reliable for continuous functions with sign changes. However, its linear convergence rate limits its efficiency in high-precision computations. In contrast, the Newton–Raphson Method demonstrates rapid quadratic convergence, offering superior speed and accuracy, although its effectiveness depends strongly on the availability of derivatives and a suitable initial approximation. The Secant Method, while derivative-free, provides a practical balance between computational efficiency and convergence speed, making it useful in situations where derivative evaluation is difficult or costly.

The numerical experiments conducted in this study confirm that all three methods converge to consistent solutions, but differ significantly in terms of iteration count, stability, and

computational cost. This variation emphasizes that no single method is universally optimal; rather, the selection of an appropriate technique depends on the nature of the function, required accuracy, and computational constraints.

Furthermore, the study indicates that modern computational practices increasingly favor hybrid and adaptive methods, which combine the stability of bracketing techniques with the efficiency of open iterative methods. Such approaches enhance convergence reliability while reducing computational effort, making them suitable for complex real-world applications in science and engineering.

In conclusion, root-finding methods remain a fundamental component of numerical analysis. A strong understanding of their theoretical foundations and practical behavior enables researchers and practitioners to select and implement the most appropriate method for solving nonlinear equations efficiently and accurately.

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